Random walks on deterministic scale-free networks: Exact results

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We study the random walk problem on a class of deterministic scale-free networks displaying a degree sequence for hubs scaling as a power law with an exponent $\gamma = \log 3/\log 2$. We find exact results concerning different first-passage phenomena and, in particular, we calculate the probability of first return to the main hub. These results allow to derive the exact analytic expression for the mean time to first reach the main hub, whose leading behavior is given by $\tau \sim V^{1-1/\gamma}$, where V denotes the size of the structure, and the mean is over a set of starting points distributed uniformly over all the other sites of the graph. Interestingly, the process turns out to be particularly efficient. We also discuss the thermodynamic limit of the structure and some local topological properties.

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I. INTRODUCTION

In the last few years, one of the most studied topics in network theory has been the investigation of highly inhomogeneous structures. Networks with strong variability in local metric and topological properties have been shown to well represent many real structures, occurring in nature and in man-made systems. Condensed matter and soft materials often feature an inhomogeneous organization in space [1], and even engineered devices can be constructed to reproduce a highly varying arrangement, in order to obtain the designed physical properties [2]. Such networks, being physically embedded in space, are finite-dimensional and it is known that in this case inhomogeneous topology can strongly affects physical phenomena occurring on the network itself.

Graphs are also used to describe the generic relation between a set of elements or agents, as it happens in complex networks theory in biology, social science, computer science, and economy [3,4]. Then, these networks often feature infinite dimensionality, and inhomogeneity has been detected in several real systems [5–8]. One of the most studied structures in the literature are scale-free (SF) networks, which show a degree sequence scaling with a characteristic power law [3,9]. This implies that highly inhomogeneous regions can be present in the network.

Random structures and stochastic approaches have appeared to be very useful in this framework, even if, in real cases, one always has to deal with a single realization of the disorder. Due to the strong variability of the topology in the samples, the quenched properties could not coincide with the behavior found in the typical cases, as described by the probability distribution. Therefore, the study of quenched samples and deterministic topologies is a very interesting task [10–15].

Apart from the discussion on how and why structures with scaling degree sequences are so often encountered in nature [16], they certainly represent an interesting class of inhomogeneous graphs. Original techniques have been developed in order to characterize these topologies and their effect on physical properties. However, the general topological and metric features of SF graphs, on the global and on the local scale, are still not completely understood.

Random walks are one of the simplest stochastic processes affected by the topology of a network and, at the same time, the basic model of diffusion phenomena and nondeterministic motion. They have been extensively studied for decades on regular lattices [17], fractal networks, and finite-dimensional inhomogeneous structures [18], where they have been shown to be able to evidence a new and unexpected phenomenon arising in presence of strong inhomogeneity, namely, the splitting between local and average properties [19]. The richer topology of a generic, inhomogeneous, and infinite dimensional graph can have a dramatic effect on the properties of random walks, especially when considering infinite graphs, which are used to describe macroscopic systems in the thermodynamic limit. Random walks represent not only a good model for diffusion phenomena on large complex networks, but also a direct way to characterize their large scale topological features, also in presence of strong inhomogeneity, and their influence on physical properties. Once these are known, they could also be used to fruitfully design and engineer a network topology with given properties.

In this paper, we want to deepen the analysis by studying random walks on a specific scale-free topology, namely, a deterministic scale-free network, built in a recursive way and featuring a scaling distribution for the hub degrees, with an exponent $\gamma = \log 3/\log 2$. For this deterministic structure, first introduced in [20], some metrical and spectral properties have recently been investigated in detail [21]. On the other hand, how these properties are linked to diffusion processes on the network is still an open problem.

Using the formalism of random walks generating functions [17], we derive exact expressions for the first-passage times [22] for a random walker starting from the maximally connected hub and from the last generation "rims" of the deterministic SF network. In particular, we investigate their dependence on the size of the network and we derive an exact expression for the mean time to first reach the most connected hub, namely, the mean time to absorption if we place a perfect trap on the most connected hub. This quantity displays an extremely slow behavior as the size of the net-

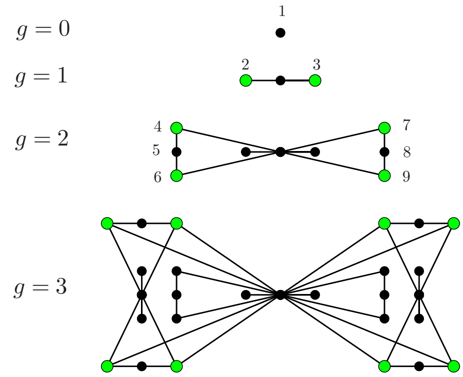

FIG. 1. (Color online) Iterative construction of the deterministic scale-free network; the first four generations are depicted. Nodes belonging to the set $\mathcal{B}_g$ are represented in brighter color.

work increases, given by $\tau \sim V^{1-1/\gamma}$, where V denotes the size of the structure, and the mean is over a set of starting points distributed uniformly over all the other sites of the graph. Interestingly, the trapping process appears to be very efficient. We also obtain an implicit relation for the generating function of the first-passage probabilities from the hub to rims, and vice versa, which provides some insight into the leading singularity for the generating function of the return probability to the hub. In particular, we prove the recurrence of the main hub by exploiting the connection between random walks and electric networks [23]. Moreover, we compare the return probability to the main hub with the return probability to an end-node evidencing a very different asymptotic behavior. All the results are checked with extensive numerical calculations.

The paper is organized as follows: in Sec. II we give a brief mathematical description of the deterministic scale-free graph, presenting the language and the formalism we will use in the whole article. Then, in Sec. III we derive the recursive relation between the first-passage probabilities from hubs to rims and vice versa; from these we calculate the exact expression for the mean time to first reach the most connected hub and its dependence on the volume of the network. In Sec. IV we calculate the return probability on the main hub and we discuss its recurrence in the thermodynamic limit. Finally, Sec. V is devoted to conclusion and discussion.

II. DETERMINISTIC SCALE-FREE NETWORK

A generic graph $\mathcal{G}$ consists in a nonempty set $\mathcal{V}$ of nodes joined pairwise by a set of links $\mathcal{L}$ [24]. Here, we consider a particular set of deterministic graphs $\{\mathcal{G}_g\}_{g=0,1,2,\ldots}$, first introduced in [20], which can be built recursively: at the gth iteration one has the graph of generation g, denoted by $\mathcal{G}_g$ (see Fig. 1).

Starting from the so-called root constituted by one single node labeled as $i=1$, at the first iteration, one introduces two more nodes $i=2,3$ and connects each of them to the root; the resulting chain of length three represents the graph of generation $g=1$. We call $\mathcal{B}_1=\{2,3\}$ the set of sites added at the first generation and linked to the root. Then, at the second iteration, one adds two chains of length three and connects each end node to the root, namely, $\mathcal{B}_2=\{4,6,7,9\}$, so that the root will increase its coordination number from 2 to 6. Proceeding analogously, at the gth iteration one introduces two replica of the existing graph of generation $g-1$ and connects the root with each site making up $\mathcal{B}_g$.

Hence, at the gth generation the root turns out to be the main hub with coordination number $2(2^g-1)$, the set of all nodes has cardinality $V_g \equiv |\mathcal{V}_g| = 3^g$, and $|\mathcal{B}_g| = 2^g$. As shown in [20], $\mathcal{G}_g$ exhibits $(2/3)3^{g-i}$ "hubs" with degree $2^{i+1}-2$, being $i \in [1, g-1]$. As a result, the tail of the degree distribution is a power-law $P(k) \sim k^{-\gamma}$ with exponent $\gamma = \log 3/\log 2 \approx 1.59$. However, this does not hold for the so-called rims contained in the sets $\mathcal{B}_g$. In this case one finds $P(k) \sim (2/3)^k = e^{-\bar{\gamma}k}$, where $\bar{\gamma} = \log(3/2) \approx 0.405$, which shows that the scaling nature of the rims is not scale-free but exponential [21]. The topological and spectral properties of this graph have been deeply analyzed in [21] where, in particular, the average degree is shown to be $\langle k \rangle_g \equiv \Sigma k P(k)/V_g = 4[1-(2/3)^g]$, namely, it approaches 4 as $g \to \infty$. Indeed, on the one hand, the graph becomes very complex and there exists a set of few $[o(1)]$ nodes whose coordination number grows indefinitely, on the other the number of nodes with degree ≤ 2 become large indefinitely; as a result the average degree remains finite, conversely the second moment $\langle k^2 \rangle$ is divergent. It should also be underlined that by increasing g the number of cycles grows fast; as a result, while the number of end-nodes grows linearly with the volume V ($\sim 3^g$), the average distance from the main hub increases only logarithmically with V ($\sim g$).

III. GENERATING FUNCTIONS AND FIRST-PASSAGE PROPERTIES

The simple random walk (RW) on a graph $\mathcal{G}$ is defined by the jumping probability p_{ij} between nearest-neighbor sites i and j

$$p_{ij} = \frac{A_{ij}}{z_i} = (\mathbf{Z}^{-1}\mathbf{A})_{ij},$$

where $Z_{ij} = z_i \delta_{ij}$ and $z_i = \Sigma_{i \in \mathcal{V}} A_{ij}$ is the coordination number of site i. Therefore, the probability of reaching in t steps site j starting from i is

$$P(i,j;t) = (p^t)_{ij}. \tag{1}$$

In the following, we denote with $P_g(i,j;t)$ the probability that a RW on $\mathcal{G}_g$, starting from a site i reaches the site with label j at time t and $F_g(i,j;t)$ the probability that the same walk reaches the site j for the first time at time t. Moreover, we consider the probability $B_g(t)$ that, at the generation g, a walk starting from any site in $\mathcal{B}_g$ first reaches the central hub and the probability $H_g(t)$ that a walk starting from the central hub first reaches any site in $\mathcal{B}_g$.

Now, the following equations hold:

$$B_g(t) = \frac{\delta_{t,1}}{g} + \frac{1}{g}\sum_{l=1}^{g-1}\sum_{k=1}^{t-1} H_l(k) B_g(t-1-k) \tag{2}$$

and

$$H_g(t) = \frac{2^{g-1}}{2^g - 1}\delta_{t,1} + \sum_{l=1}^{g-1}\sum_{k=1}^{t-1}\frac{2^{l-1}}{2^g-1}B_l(k)H_g(t-1-k). \quad (3)$$

Some comments are in order here. In a graph of generation g, each rim is connected to g nodes of which one is the main hub; this accounts for the first term in Eq. (2). The remaining $g-1$ links connect each rim to a minor hub from which one can reach the main hub only passing through a rim; this explains the sum of convolutions in Eq. (2). Analogously, from the main hub 2^g links out of $2(2^g-1)$ point directly to a rim; the remaining links connect the main hub to nodes corresponding to rims of graphs of generations $l<g$ and from such nodes one can reach any site in $\mathcal{B}_g$ only through the main hub itself.

The generating functions corresponding to Eqs. (2) and (3) read as

$$\widetilde{B}_g(z) \equiv \sum_{t=0}^{\infty} B_g(t)z^t = \frac{z}{g} + \frac{z}{g}\sum_{l=1}^{g-1}\widetilde{H}_l(z)\widetilde{B}_g(z) \quad (4)$$

and

$$\widetilde{H}_g(z) \equiv \sum_{t=0}^{\infty} H_g(t)z^t = \frac{2^{g-1}}{2^g-1}z + \frac{\widetilde{H}_g(z)}{2^g-1}z\sum_{l=1}^{g-1}2^{l-1}\widetilde{B}_l(z). \quad (5)$$

Notice that, as can be easily inferred from Fig. 1, $B_1(t) = H_1(t) = \delta_{t,1}$, from which $\widetilde{B}_1(z) = \widetilde{H}_1(z) = z$. We also recall that, by definition, $\widetilde{B}_g(1)$ is just the probability to ever reach the hub from any site in $\mathcal{B}_g$ and, analogously, $\widetilde{H}_g(1)$ is the probability to ever reach any site in $\mathcal{B}_g$ from $i=1$. Of course, for finite g, one has $\widetilde{B}_g(1) = \widetilde{H}_g(1) = 1$. With some algebra we rewrite Eqs. (4) and (5), respectively, as

$$\widetilde{B}_g(z)\left(\frac{g}{z} - \sum_{l=1}^{g-1}\widetilde{H}_l(z)\right) = 1 \quad (6)$$

$$\widetilde{H}_g(z)\left(\frac{2^g-1}{2^{g-1}z} - \sum_{l=1}^{g-1}2^{l-g}\widetilde{B}_l(z)\right) = 1. \quad (7)$$

Moreover, by properly handling Eqs. (6) and (7) we can obtain the following two finite-difference equations coupled together:

$$\frac{1}{\widetilde{B}_{g+1}(z)} - \frac{1}{\widetilde{B}_g(z)} = \frac{1}{z} - \widetilde{H}_g(z) \quad (8)$$

$$\widetilde{B}_g(z) - \frac{2}{z} = -\frac{2}{\widetilde{H}_{g+1}(z)} + \frac{1}{\widetilde{H}_g(z)}, \quad (9)$$

which can be combined together to get the rather symmetric expression $\widetilde{B}_g(z)\{[\widetilde{B}_{g+1}(z)]^{-1} - [\widetilde{B}_1(z)]^{-1}\} = 2\widetilde{H}_g(z)\{[\widetilde{H}_{g+1}(z)]^{-1} - [\widetilde{H}_1(z)]^{-1}\}$.

It is interesting to notice that, since $0 \le \widetilde{H}_g(z) \le 1$, for any g and for any z, from Eq. (6) we have $z/g \le \widetilde{B}_g(z) \le z/[g(1-z)+z]$, from which one gets $\widetilde{B}_\infty(z) = 0$. Hence, in the thermodynamic limit the probability to eventually reach the hub from $\mathcal{B}_g$ is zero.

From Eqs. (6) and (7) it is possible to calculate recursively $\widetilde{B}_g(z)$ and $\widetilde{H}_g(z)$; for instance, for the first generations we get

$$\widetilde{B}_2(z) = \frac{z}{2-z^2}, \quad (10)$$

$$\widetilde{B}_3(z) = \frac{z(3-z^2)}{9-8z^2+z^4}, \quad (11)$$

$$\widetilde{B}_4(z) = \frac{z(3-z^2)(14-11z^2+z^4)}{168-282z^2+145z^4-24z^6+z^8}, \quad (12)$$

and

$$\widetilde{H}_2(z) = \frac{2z}{3-z^2}, \quad (13)$$

$$\widetilde{H}_3(z) = \frac{4z(2-z^2)}{14-11z^2+z^4}, \quad (14)$$

$$\widetilde{H}_4(z) = \frac{8z(2-z^2)(9-8z^2+z^4)}{270-435z^2+211z^4-31z^6+z^8}. \quad (15)$$

The generating functions $\widetilde{B}_g(z)$ and $\widetilde{H}_g(z)$ allow to calculate the average time t_g^B taken by a random walk started on a site in $\mathcal{B}_g$ to reach the hub and the average time t_g^H taken by a random walk started on the main hub to reach any site in $\mathcal{B}_g$, respectively. In fact, for any arbitrary generation g we can write

$$t_g^B = \frac{\partial}{\partial z}\widetilde{B}_g(z)|_{z=1} \quad (16)$$

$$t_g^H = \frac{\partial}{\partial z}\widetilde{H}_g(z)|_{z=1}. \quad (17)$$

Hence, deriving Eqs. (6) and (7) and recalling that for finite structures $\widetilde{B}_g(1) = \widetilde{H}_g(1) = 1$, we find the following coupled equations:

$$t_g^B = g + \sum_{l=1}^{g-1} t_l^H, \quad (18)$$

$$t_g^H = 2 - 2^{1-g} + 2^{-g}\sum_{l=1}^{g-1} 2^l t_l^B. \quad (19)$$

Notice that the previous two equations could be obtained directly from Eqs. (2) and (3) by multiplying both sides by t and summing over $t=0,\ldots,\infty$.

Now, from Eqs. (18) and (19) we get

$$t_{g+1}^B - t_g^B = 1 + t_g^H \quad (20)$$

$$2t_{g+1}^{H} - t_{g}^{H} = 2 + t_{g}^{B}. \tag{21}$$

Such equations can be properly handled to get

$$2t_{g+2}^{H} - 3t_{H}^{g+1} = 1, \tag{22}$$

and

$$2t_{g+2}^{B} - 3t_{g+1}^{B} = 3, \tag{23}$$

whose solution is given by

$$t_{g}^{H} = \frac{4}{3}\left(\frac{3}{2}\right)^{g} - 1, \tag{24}$$

and

$$t_{g}^{B} = \frac{8}{3}\left(\frac{3}{2}\right)^{g} - 3, \tag{25}$$

where we used $t_1^H = t_1^B = 1$.

Interestingly, t_g^B and t_g^H both display the same exponential law: the average time to first reach the hub from any of the 2^g sites making up $\mathcal{B}_g$ and the average time to first reach any site in $\mathcal{B}_g$ from the hub grows with the generation as $\sim(3/2)^g$. Also, recalling that $|\mathcal{B}_g| = 2^g$ and that $V_g = 3^g$, we can write $2^g = V^{\log 2/\log 3}$ and get $(3/2)^g = V_g/|\mathcal{B}_g| = V^{1-\log 2/\log 3}$.

The average times t_g^B and t_g^H found above are useful to calculate the mean time to absorption τ_g. In fact, let us define $t_g(i)$ the mean time necessary to first reach the hub from site $i \neq 1$, then $\tau_g \equiv \sum_{i \in \mathcal{V}_g'} t_g(i)/(V_g - 1)$ and we can write

$$\sum_{i \in \mathcal{V}_g'} t_g(i) = \sum_{i \in \mathcal{V}_{g-1}'} t_{g-1}(i) + \sum_{i \in \mathcal{V}_g \setminus \mathcal{V}_{g-1}} t_g(i), \tag{26}$$

namely,

$$\tau_g = \tau_{g-1}\frac{V_{g-1} - 1}{V_g - 1} + \frac{1}{V_g - 1}\sum_{i \in \mathcal{V}_g \setminus \mathcal{V}_{g-1}} t_g(i), \tag{27}$$

where $\mathcal{V}_g' \equiv \mathcal{V}_g \setminus \{1\}$ and $\mathcal{V}_g \setminus \mathcal{V}_{g-1}$ is the set of sites added at the gth generation. We now notice that the last sum in Eq. (27) is just given by

$$\sum_{i \in \mathcal{V}_g \setminus \mathcal{V}_{g-1}} t_g(i) = t_g^B|\mathcal{B}_g| + (t_g^B + 1)\frac{|\mathcal{B}_g|}{2} + \sum_{l=1}^{g-2}[\tau_g(3^l - 1) + 3^l t_{l+1}^H + 3^l t_g^B]2^{g-l-1}, \tag{28}$$

where the first two terms simply allow for $|\mathcal{B}_g|$ rims and $|\mathcal{B}_g|/2$ nodes bridging between two rims; then, from the remaining nodes making up $\mathcal{V}_g \setminus \mathcal{V}_{g-1}$ one can reach the main hub by passing through any rim, possibly via a minor hub, and this yields to the last sum.

By replacing in the previous equation the expressions for t_g^B and t_g^H found in Eqs. (24) and (25), we get

$$\sum_{i \in \mathcal{V}_g \setminus \mathcal{V}_{g-1}} t_g(i) = \frac{2^g}{5} + \frac{32}{15}\left(\frac{9}{2}\right)^g - \frac{8}{3}3^g + 2^{g-1}\sum_{l=1}^{g-2}\tau_l\frac{3^l - 1}{2^l}, \tag{29}$$

which, together with Eq. (27) yields

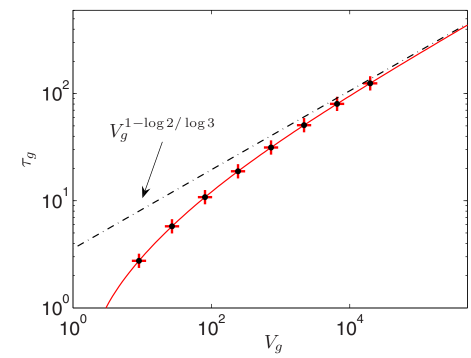

FIG. 2. (Color online) Mean first-passage time τ^g for a simple random walker moving on the deterministic sale-free network as a function of the volume V_g; data from the exact analytic expression in Eq. (30) (●) are compared to the asymptotic form of Eq. (31) (dash-dotted line) and to the numerical solution of Eq. (33) (+, the continuous line is a guide to the eye).

$$(3^{g+1} - 1)\tau^{g+1} - 3(3^g - 1)\tau^g = \frac{16}{3} \times 3^g\left[\left(\frac{3}{2}\right)^g - \frac{1}{2}\right].$$

The solution of this recursive equation is

$$\tau_g = \frac{1}{3^g - 1}\left[\frac{32}{9}\left(\frac{9}{2}\right)^g - \frac{2}{9}(17 + 4g)3^g\right], \tag{30}$$

whose leading behavior is given by

$$\tau^g \sim V_g^{1-\log 2/\log 3} = V_g^{1-1/\gamma}, \tag{31}$$

with $1 - \log 2/\log 3 \approx 0.37$. Such an exponent is even lower than the exponent $\log 2/\log 3 = 1/(\gamma' - 1) \approx 0.63$ found in [10] for a deterministic scale-free network displaying the power degree $\gamma' = 1 + \log 3/\log 2 = 1 + \gamma$. Indeed, the multifractal nature displayed by the graph under study gives rise to a remarkable efficiency for trapping on $i = 1$.

Numerical calculations

For a generic graph, given the corresponding adjacency matrix $\mathbf{A}$ and the coordination matrix $\mathbf{Z}$, the numerical calculation of the mean time to absorption can be performed by exploiting a differential equation where the normalized discrete Laplacian $\mathbf{\Delta} = \mathbf{A}\mathbf{Z}^{-1} - \mathbf{I}$ appears [10,11,25,26]. More precisely, for the topological structures analyzed here, the Laplacian $\mathbf{\Delta}_g$ is a $V_g \times V_g$ matrix which depends on the generation g and we have

$$-\sum_{j=2}^{V_g} \mathbf{\Delta}_{\mathbf{g}_{ij}} t_g(j) = 1. \tag{32}$$

Therefore, the mean time to absorption averaged over all possible starting sites $i \neq 1$ reads as

$$\tau_g = \frac{1}{V_g - 1}\sum_{i=2}^{V_g} t_g(i) = \frac{1}{V_g - 1}\sum_{i=2}^{V_g}\sum_{j=2}^{V_g}(-\mathbf{\Delta}_{\mathbf{g}}^{-1})_{ij}. \tag{33}$$

In Fig. 2 we compare the analytical results of Eq. (30) and of Eq. (31) with the numerical results obtained via Eq. (33).

IV. RETURN PROBABILITY

The results found in the previous section allow to deepen the analysis of the random walk problem on the determinist structure considered. In particular, the probability $F_g(1,1;t)$ that a random walk started on the hub returns to the hub itself for the first time after time t has the form

$$F_g(1,1;t) = \sum_{i=1}^{g} \frac{|\mathcal{B}_i|}{\mathbf{Z}_{\mathbf{g}_{11}}} B_i(t-1) = \sum_{i=1}^{g} \frac{2^i}{2(2^g-1)} B_i(t-1) \tag{34}$$

and the pertaining generating function is

$$\tilde{F}_g(z) \equiv \sum_{t=0}^{\infty} F_g(1,1;t) z^t = z \sum_{i=1}^{g} \frac{2^{i-1}}{2^g-1} \tilde{B}_i(z). \tag{35}$$

By replacing the expression for $\tilde{B}_g(z)$ appearing in Eq. (9), we get the telescopic sum $\Sigma_{i=1}^{g}[2^{i+1}/\tilde{H}_{i+1}(z) - 2^i/\tilde{H}_i(z)]$, so that Eq. (35) simplifies into

$$\begin{aligned}\tilde{F}_g(z) &= 2 + \frac{z}{2(2^g-1)}\left[\frac{2}{\tilde{H}_1(z)} - \frac{2^{g+1}}{\tilde{H}_{g+1}(z)}\right] \\ &= 2 + \frac{1}{2^g-1} - \frac{2^g z}{(2^g-1)\tilde{H}_{g+1}(z)},\end{aligned} \tag{36}$$

which highlights that the probability to first return on the hub at generation g directly depends on the probability to first reach any site in $\mathcal{B}_{g+1}$ starting from $i=1$. We also notice that for finite g, $\tilde{F}_g(1)=1$, that is the hub is a recurrent point, as expected for any point on finite graphs [19].

Interestingly, from Eq. (34) we can explicitly derive the average time t_g^O to first return to the main hub as a function of t_g^H

$$t_g^O = \frac{\partial}{\partial z} \tilde{F}_g(z)|_{z=1} = \frac{t_{g+1}^H - 1}{1-2^{-g}} = \frac{2(3^g-2^g)}{2^g-1}, \tag{37}$$

where in the last equality we used the result of Eq. (24). Therefore, for large structures we have $t_g^O \sim (3/2)^g$, which is the same leading behavior found for t_g^B and t_g^H. This result is also consistent with Kac formula [27] according to which $t_g^O = [P(1,1;\infty)]^{-1} = 2|\mathcal{L}_g|/\mathbf{Z}_{\mathbf{g}_{11}} = \Sigma_{i \in \mathcal{V}_g} \mathbf{Z}_{\mathbf{g}_{ii}}/\mathbf{Z}_{\mathbf{g}_{11}}$.

From $\tilde{F}_g(z)$ we can now calculate $\tilde{P}_g(z) \equiv \Sigma_{t=0}^{\infty} P_g(1,1;t) z^t = [1-\tilde{F}_g(z)]^{-1}$ [19]: using Eqs. (6) and (36), we get

$$\tilde{P}_g(z) = \frac{(2^g-1)\tilde{H}_{g+1}(z)}{2^g[z - \tilde{H}_{g+1}(z)]}. \tag{38}$$

From Eqs. (13)–(15), we can derive

$$\tilde{P}_1(z) = \frac{1}{2}\frac{2}{1-z^2}, \tag{39}$$

$$\tilde{P}_2(z) = \frac{3}{4}\frac{4(2-z^2)}{(1-z^2)(6-z^2)}, \tag{40}$$

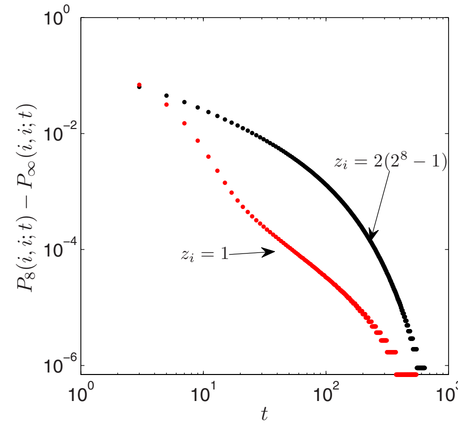

FIG. 3. (Color online) Normalized return probability $P_g(i,i;t) - P_\infty(i,i;t)$ for a random walk on $\mathcal{G}_8$ and started at the main hub ($i=1$) and at an end-node belonging to $i \in \mathcal{V}_g \setminus \mathcal{V}_{g-1}$, as a function of time t; only even instants of time have been depicted.

$$\tilde{P}_3(z) = \frac{15}{16}\frac{8(2-z^2)(9-8z^2+z^4)}{(1-z^2)(126-109z^2+22z^4-z^6)}. \tag{41}$$

We checked up to generation $g=8$ that $\tilde{P}_g(z)$ can be written as

$$\tilde{P}_g(z) = (2^g-1)\frac{f(z)}{(1-z^2)g(z)}, \tag{42}$$

where $f(z)$ and $g(z)$ are even polynomial of degree 2^g-2 (in fact all possible cycles have even length), both devoid of any factor $(1-z^2)$ and satisfying $f(1)/g(1) = 1/(3^g-2^g)$. In Fig. 3 we show data for $P_8(1,1;t)$ and for $P_8(i,i;t)$, being $i \in \mathcal{V}_g \setminus \mathcal{V}_{g-1}$ and $z_i=1$, both obtained numerically from Eq. (1): $P_g(i,i;t) = [(\mathbf{\Delta_g} + \mathbf{I})^t]_{ii}$. Interestingly, the decay displayed by the two probabilities considered is significantly different, which provides a further evidence of the strong inhomogeneity of the graph.

Once the thermodynamic limit is taken, a divergence in $\tilde{P}_g(z)$ for $z \to 1$ would imply that the main hub is recurrent [19]. Recurrence can alternatively be proven by exploiting the connection between random walks an electric networks. The escape probability can be determined by calculating the effective resistivity R_{eff} at fixed distance from the hub, when all the links of the network are replaced by unit resistors and a unit voltage is applied between the hub and the points at fixed distance from it. Then, taking the thermodynamic limit, the relation between the escape probability from the hub and the effective resistivity of the network reads [23]

$$P_{esc} = \lim_{g \to \infty} \frac{1}{\mathbf{Z}_{\mathbf{g}_{11}} R_{eff}(g)}, \tag{43}$$

where, we recall, $\mathbf{Z}_{\mathbf{g}_{11}} = 2(2^g-1)$ is the coordination number of the hub at generation g. Therefore, if $\mathbf{Z}_{\mathbf{g}_{11}} R_{eff}(g)$ diverges in the thermodynamic limit, the hub is recurrent, while if it converges to a finite value, there is a finite probability of escape from the hub. On highly inhomogeneous graphs such as our scale-free networks, however, a vanishing P_{esc} may not necessarily imply that the mean number of visits to the hub is infinite [28].

In general, when the unit voltage is applied to the resulting circuit, by symmetry one can detect nodes which are at the same voltage (e.g., the rims) and short them together without affecting the distribution of currents in the branches, nor the effective resistivity. In this way, by applying the standard rules for the sum of resistors in series and in parallel, one can build an equivalent network for the circuit, in terms of the total resistivity. To better exploit the symmetry properties of the network, we consider only even values of g, so that the points at a maximum distance from the hub are topologically equivalent (an analogous relation can be easily obtained if only odd values of g are considered). With some algebra one then obtains

$$R_{eff}(g) = \frac{1}{2^{g/2}} \sum_{k=0}^{g/2-1} \frac{1}{2^k} = \frac{1}{2^{g/2-1}}\left(1 - \frac{1}{2^{g/2}}\right) \qquad (44)$$

holding for even g. Hence, recalling that $\mathbf{Z}_{\mathbf{g}_{11}}=2(2^g-1)$ we derive that the main hub is recurrent.

V. CONCLUSION AND DISCUSSION

In this work we studied the random walk problem on a class of deterministic networks exhibiting both a scale-free, $P(k)\sim k^{-\gamma}$, and an exponential, $P(k)\sim(2/3)^k$, degree distribution. The latter holds for a subset of nodes called rims which are at a distance 1 from the main node. Adopting a generating function formalism we calculate the exact average time τ to first reach the hub, where the mean is taken over all possible walks connecting a site i to the hub and over all starting nodes i. The leading behavior for τ turns out to be $\tau\sim V^{1-1/\gamma}$, where V is the total number of sites making up the network. Analogous power-law behaviors were found previously for exactly decimable fractals [25,26] as well as for other deterministic scale-free networks [10] and Apollonian networks [11], however, the structure considered here results remarkably effective, being $1-1/\gamma\approx 0.37$. The reason lays in the large number of cycles and in the multifractal nature of the graph under study which determines a very short average distance from the main hub.

In the second part of the work, we focused on the probability to first return to the main hub and we obtained a recursive formula for its generating function which allows to get some insight into the local topological properties of the hub. In particular, the recurrence of the hub is proved by mapping the graph into an electrical network whose effective resistivity provides the escape probability from the hub. Hence, in the thermodynamic limit, a splitting between average and local properties occurs: transience in the average is expected due to the diverging dimensionality, while the hub is locally recurrent.

Finally, we compared the return probability to the main hub with the return probability to one of the outer nodes evidencing different asymptotic behaviors. This further highlights the strong inhomogeneity of the graph considered.

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